Module C Comprehensive

- Verify the on-line record [the grades page]
- No review period
  - But encouraged to discuss it with the instructor
  No negotiation
- Return the package today
  - May take home if returned by coming Tue.
  - Available for review in the instructor's office
- Mini Project Phase 2
  - Review and include it in the Final Package

Unit D2: Relations

Today

- Understand and use equivalence relation, equivalence class, and partition
- Prove certain properties of relations, etc.
- Take-home exercises
  - Space stations, Bogus proofs & paradoxes

Section 1

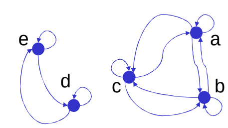
Example Relations

- Weeding
  - Relation: "is connected to (through the root system)"
- Academic integrity
  - Relation: "shares information with"
- Room temperature (temp. zones)
  - Relation: "has opening to"
- Flash Psychic

Properties?

Equivalence Relation

- Relation with all of the following properties:
  - Reflexive
  - Symmetric
  - Transitive

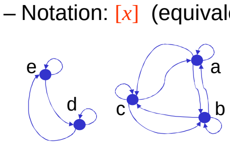


- Cf. the relation in a poset

What can you say about the set members?

Equivalence Class

- A subset that are mutually related through an equivalence relation
  - Notation: $[x]$ (equivalence class including x)

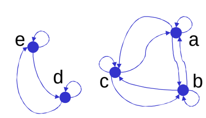


$[a] = [b] = [c] = \{a, b, c\}$
$[d] = [e] = \{d, e\}$

- (more formally) For an equivalence relation $R \subseteq A \times A$, X is an equivalence class if $X \subseteq A$ and $\forall x, y \in X\ ((x, y) \in R)$

Partition

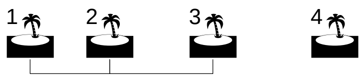
- The set of equivalence classes



$\{\{a, b, c\}, \{d, e\}\} = \{[a], [d]\}$

- Properties
  - The subsets are all disjoint.
  - The union of the subsets covers the entire set.
- (more formally) For an equivalence relation $R \subseteq A \times A$, the partition of A is defined as $\{[x] \mid x \in A\}$

Group Exercise 1: Weeding



- Define the equivalence relation
- Find all equivalence classes
- Find the partition

Group Exercise 2: Flash Psychic

- Informally describe the involved equivalent relation *R*

between the numbers on the number-symbol correspondence charts

- Formally define the partition induced by *R*

Section Summary

- Equivalence relation
  - Reflexive, symmetric, transitive relation
- Equivalence relation forms equivalence classes.
- Equivalence relation induces a partition

Section 2

Mathemagic

- Question: Is 0.999... equal to 1.0?

Proof?

Review: B5

Proof Structure

Hypothesis: ***o*** ® ***f***, ***o***

Proof of ***f*** from the above hypothesis

1. ***o*** ® ***f*** [hyp]
2. ***o*** [hyp]
3. ***f*** [MP: 1, 2]

proof

conclusion/theorem

justification for each step (must be semantically correct)

line numbering

Generalized Flash Psychic

- Hypotheses
  - *m* is a positive integer.
  - $R = \{(x, y) \mid (x \bmod m) = (y \bmod m)\}$
- Conclusion:

  R is an equivalence relation.

Example members?

- Note: *x* mod *y* is the remainder of *x* divided by *y*.

Cf. FSA to recognize 10, 010, 110, 0010, 0110, 1010, 1110

Proof

1. R is reflexive. [Lemma 1]
2. R is symmetric. [Lemma 2]
3. R is transitive. [Lemma 3]
4. R is an equivalence relation. [Def. eq. rel. 1.-3.]

Note: Lemma = preliminary theorem

Lemma 1: R is reflexive.

Hypotheses

- m is a positive integer.
- $R = \{(x, y) \mid (x \bmod m) = (y \bmod m)\}$

Conclusion: For any x, $(x, x) \in R$

1. $(x \bmod m) = (x \bmod m)$ [Def. function]
2. R is reflexive. [Def. reflexive]

"trivial," "obvious," "easy," etc.?

Lemma 2: R is symmetric.

Def: For any x, y, if $(x, y) \in R$, then $(y, x) \in R$

Hypotheses

- m is a positive integer.
- $R = \{(x, y) \mid (x \bmod m) = (y \bmod m)\}$
- $(x, y) \in R$

Conclusion: $(y, x) \in R$

1. $(x \bmod m) = (y \bmod m)$ [Hyp.]
2. $(y \bmod m) = (x \bmod m)$ [__________________]
3. R is symmetric. [Def. symmetric]

Lemma 3: R is transitive.

Def: For any x, y, z, if (x, y), $(y, z) \in R$, then $(x, z) \in R$

Hypotheses

- m is a positive integer.
- $R = \{(x, y) \mid (x \bmod m) = (y \bmod m)\}$
- (x, y), $(y, z) \in R$

Conclusion: $(x, z) \in R$

1. $(x \bmod m) = (y \bmod m)$ [Hyp.]
2. $(y \bmod m) = (z \bmod m)$ [Hyp.]
3. __________________ [__________________]
4. R is transitive. [Def. transitive]

Prove

- There is a country where the breakfast is served with boiled eggs cooked for either 5 or 8 minutes.

Disprove

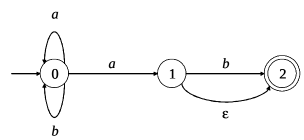
- There is a unicorn in this room.

Prove or Disprove

- There is a cave system whose total length reaches hundreds of miles.
  - Imagine a tunnel from Philadelphia to, say, Boston

Prove or Disprove

- Nondeterminism can be simulated as determinism applying the idea of "power set."
- Note: Try this in terms of FSA's



Proof/Disproof Strategy

- To prove a general statement, follow the proof pattern discussed in Unit B5. That is, identify the hypotheses (if any) and the conclusion, number proof steps, and give justification for each step.
- To disprove a general statement, give a counterexample.
- To prove an existential statement, give an example [proof by existence/example].
- To disprove an existential statement, follow the proof pattern as noted above and justify that there is ***no*** such thing.
- If not sure to prove/disprove, try both.

Review: B5

Proof Methods Summary

- Direct proof: Straightforward proof of a conclusion from hypotheses (if any)
  - Proof by cases: Prove exclusive and exhaustive cases separately
- Indirect proof: Uses a proof of a proposition different from the conclusion
  - Proof by contradiction: Assume the negation of the conclusion and derive a contradiction

Section Summary

- Roles of logic
  - To specify structures
  - To justify by proving or disproving (semantically correct syntactic manipulation)
- Approaches to proof
  - General statement (formally, $\forall x$ ...)
    - Prove: Cover all possibilities
    - Disprove: Counterexample
  - Existential statement (formally, $\exists x$ ...)
    - Prove: Example
    - Disprove: Cover all impossibilities

Summary Exercise

- Hypothesis: Direct or indirect interaction between objects in the universe is an equivalence relation. Make up your own universe.
- Prove or disprove: The universe is a partition (induced by the above equivalence relation) with a cardinality greater than 1.
  - If difficult, explain why (definition? proof step?).
- [optional] Philosophical implications of your answer?
- Questions/Comments/Suggestions